Linear Independence

Department of Computer Engineering

Sharif University of Technology

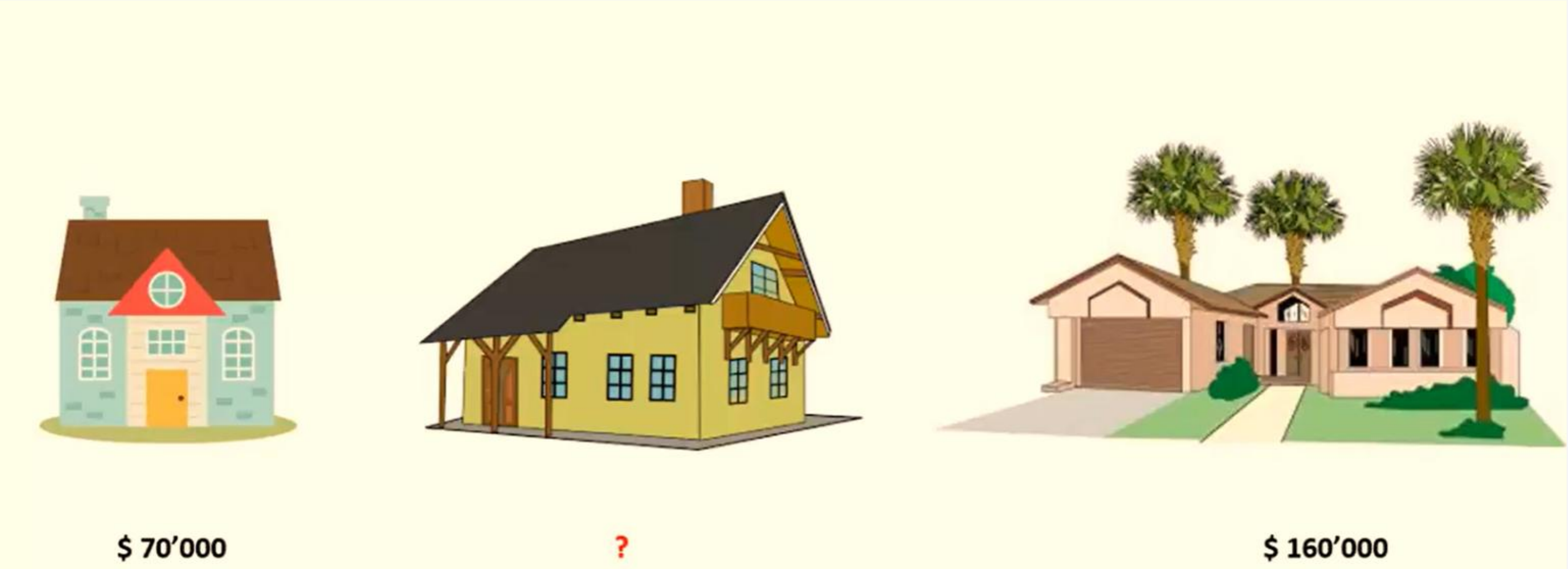
Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

Table of contents

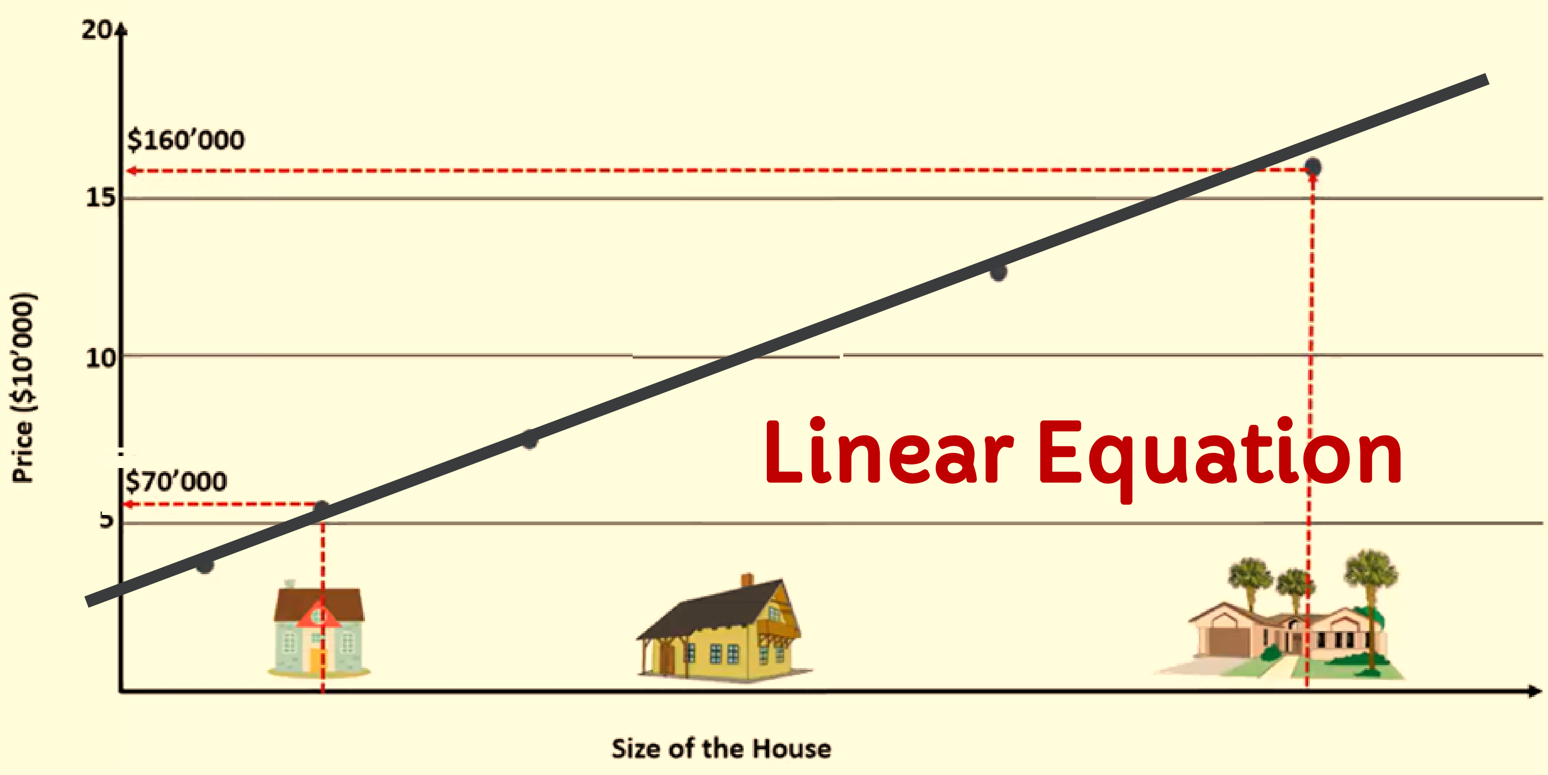
01

Introduction

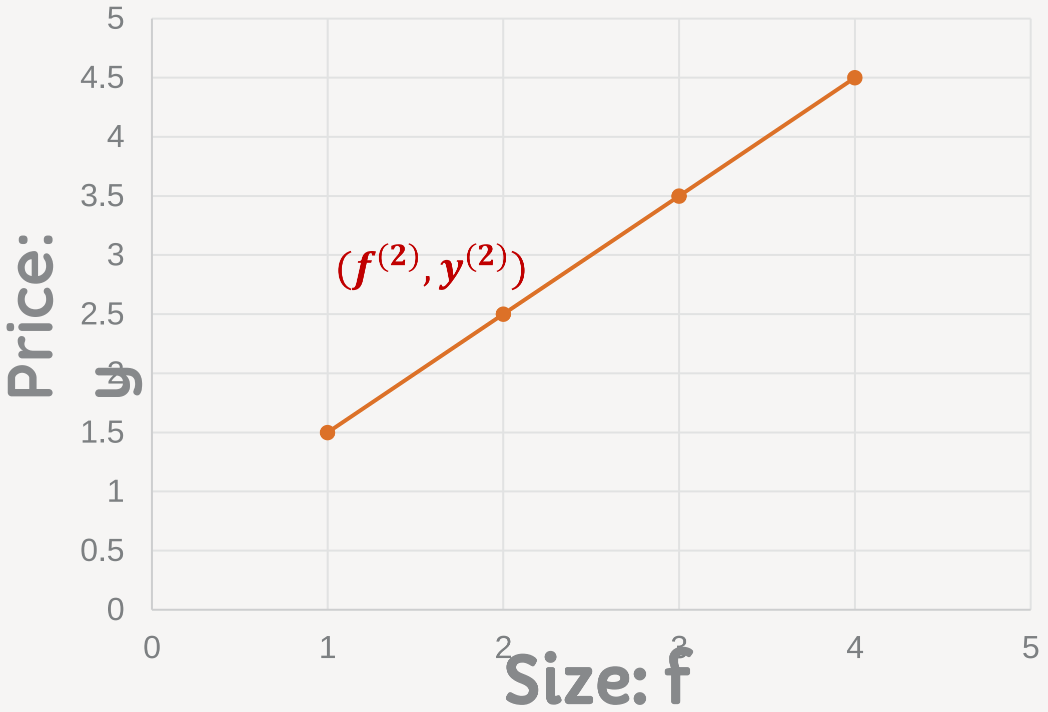
Price Problem



Price Problem



Linear Equation with offset



$$y = af + c$$

How to convert it to matrix-vector multiplication?

$$Ax = b$$

House Features

- #Room
- Size
- #Bedroom
- Age
- Address features: Street, Alley, …
- Size of part1, part2, part3, part4
- Floors
- #Bathrooms
- …

Which features are dependent on others?

02

Linear Independence

Linear Independence (Algebra)

Definition

Dependent

- For at least one $\lambda \neq 0$ $$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \qquad \lambda \in \mathbb{R}$$
- A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.

Linear Independence (Algebra)

Definition

Independent

- Only when all $\lambda_i = 0$ $\quad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \quad \lambda \in \mathbb{R}$
- No vector in the set is a linear combination of the others (has only the trivial solution)

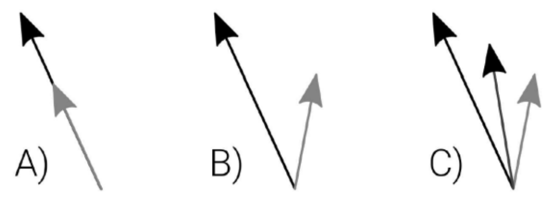
Linear Independence (Geometry)

Definition

A set of vectors is linear independent if the subspace dimensionality (its span) equals the number of vectors.

Example

- Are these vectors linearly independent?



Geometric sets of vectors in $\mathbb{R}^2$

Example

Example

- Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, and\ v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- a) $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b) $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Properties

Theorem 1

Any set of vectors that contains the zeros vector is guaranteed to be linearly dependent.

Characterization of Linearly Dependent sets

Theorem 2

An indexed set $S = \{v_1, \ldots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then **some** v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Proof

Notes!!!

- Does not say that every vector
- Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Characterization of Linearly Dependent sets

Proof

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1 v_1 = 0$, which is impossible because $v_1 \neq 0$. So $j > 1$ and

$$c_1 v_1 + \cdots + c_j v_j + 0 v_{j+1} + 0 v_n = 0$$

$$c_j v_j = -c_1 v_1 - \cdots - c_{j-1} v_{j-1}$$

$$v_j = \left(-\frac{c_1}{c_j}\right) v_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right) v_{j-1}$$

Characterization of Linearly Dependent sets

Proof

If some v_j in S equals a linear combination of the other vectors, then v_j can be subtracted from both sides of the equation, Producing a linear dependence relation with a nonzero weight (-1) on v_j. [For instance, if $v_1 = c_2v_2 + c_3v_3$, then $0 = (-1)v_1 + c_2v_2 + c_3v_3 + 0v_4 + \cdots + 0v_n$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If v_1 is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $v_1 \neq 0$, and there exist weights $c_1, \dots, c_n$ not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_pv_n = 0$$

Properties

- The vectors coming from the vector form of the solution of a matrix equation Ax = 0 are linearly independent

Example

- Vectors related to x_2 and x_3 are linear independent.
- Columns of A related to to x_2 and x_3 are linear dependent.
- Span$\{A_1, A_2, A_3\} = Span\{A_1\}$

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Properties

Important

- If a collection of vectors is linearly dependent, then any superset of it is linearly dependent.
- Any nonempty subset of a linearly independent collection of vectors is linearly independent.

Properties

Theorem 3

- ❑ Any set of $\mathrm{p} > n$ vectors in $\mathbb{R}^n$ is necessarily dependent.
- ❑ Any set of $\mathrm{p} \leq n$ vectors in $\mathbb{R}^n$ could be linearly independent.

Proof

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

Exercise

Example

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Linear Dependent Properties

- Suppose vectors $v_1, \dots, v_n$ are linearly dependent:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1 \neq 0$. Then:

$$span\{v_1, \dots, v_n\} = span\{v_2, \dots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

Linear combinations of linearly independent vectors

Theorem 4

Suppose x is linear combination of linearly independent vectors

$v_1, \dots, v_n$:

$$x = \beta_1 v_1 + \cdots + \beta_n v_n$$

The coefficients $\beta_1, \dots, \beta_n$ are unique.

Proof

Conclusion

- Step 1: Count the number of vectors (call that number p) in the set and compare to n in $\mathbb{R}^n$. As mentioned earlier, if $\mathbf{p} > n$, then the set is necessarily dependent. If $\mathbf{p} \leq n$ then you have to move on to step 2.
- Step 2: Check for a vector of all zeros. Any set that contains the zeros vector is a dependent set.
- The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.

03

Functions Linearly Independent

Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called linearly dependent if there are nonzero constants c_1 and c_2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called linearly independent.

Example

Linearly dependent or independent?

- Functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - cos^2 t$
- Functions $\{\sin^2 x, \cos^2 x, \cos(2x)\} \subset \mathcal{F}$

04

Polynomials Linearly Independent

Vector Space of Polynomials

Example

Are $(1-x), (1+x), x^2$ linearly independent?

Linearly Independent Sets versus Spanning Sets

Span	Linearly Independent
Want many vectors in small space	Want few vectors in big space
Adding vectors to list only helps	Deleting vectors from list only helps
Suppose that $v_1, \dots, v_k$ are columns of A, now we have: AX= b has solution $\Leftrightarrow b \in span\{v_1, \dots, v_k\}$	Suppose that $v_1, \dots, v_k$ are columns of A, now we have: AX = 0 has only trivial solution(X=0) $\Leftrightarrow v_1, \dots, v_k$ are linearly independent.

Resources

- Page 97 LINEAR ALGEBRA: Theory, Intuition, Code
- Page 213: David Cherney,
- Page 54: Linear Algebra and Optimization for Machine Learning